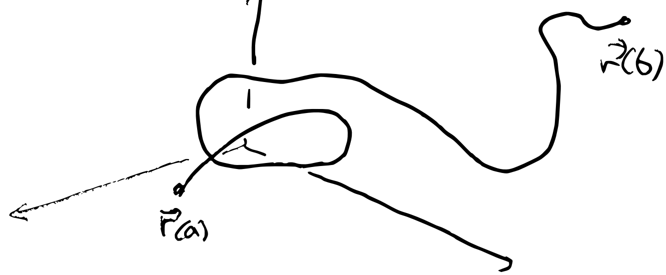
Arc length

Now we can start doing "calculus" for parametric curves. Given a parametric curve $\vec{r}(t)$, $a \le t \le b$,

what is the "length" of the curve?



$$\boxed{\text{Arclength} = \int_a^b |\vec{r}\,'(t)|\,dt}$$

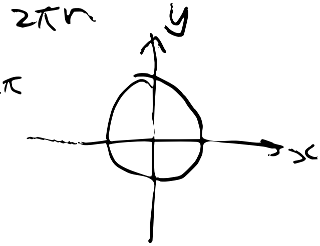
This makes sense, because you are adding the infinitesimal lengths to get the total length, the arclength.

$\vec{r}(b) - \vec{r}(a) =$ "the total vector" $= \int_a^b \vec{r}\,'(t)\,dt$ (vector)

arclength = "the total length" $= \int_a^b |\vec{r}\,'(t)|\,dt$ (scalar)

Example Let's check that the circumference of a circle with radius r is $2\pi r$

$\vec{r}(t) = \langle r\cos t, r\sin t \rangle$, $0 \le t \le 2\pi$



$\Rightarrow \vec{r}\,'(t) = \langle -r\sin t, r\cos t \rangle$

$\Rightarrow |\vec{r}\,'(t)| = \sqrt{(-r\sin t)^2 + (r\cos t)^2}$

$= \sqrt{r^2\sin^2 t + r^2\cos^2 t} = \sqrt{r^2} = r$

$\Rightarrow$ circumference $= \int_0^{2\pi} r\,dt = 2\pi r$.

One interesting aspect of this formula is that the formula gives the arclength regardless of the parametrization.

Example We can chose a weird parametrization

$\vec{r}(t) = \langle r\cos(t^2), r\sin(t^2) \rangle$, $0 \le t \le \sqrt{2\pi}$, which still represents a circle.

$\Rightarrow \vec{r}\,'(t) = \langle -2rt\sin(t^2), 2rt\cos(t^2) \rangle$

$\Rightarrow |\vec{r}\,'(t)| = \sqrt{(-2rt\sin(t^2))^2 + (2rt\cos(t^2))^2} = \sqrt{4r^2t^2(\sin^2(t^2) + \cos^2(t^2))}$

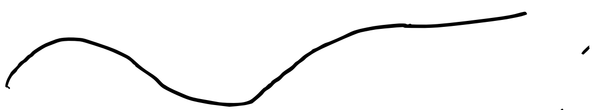
$= \sqrt{4r^2t^2} = 2r|t|$, and

arclength $= \int_0^{\sqrt{2\pi}} 2r|t|\,dt = \int_0^{\sqrt{2\pi}} 2rt\,dt = rt^2\Big|_0^{\sqrt{2\pi}} = 2\pi r$!

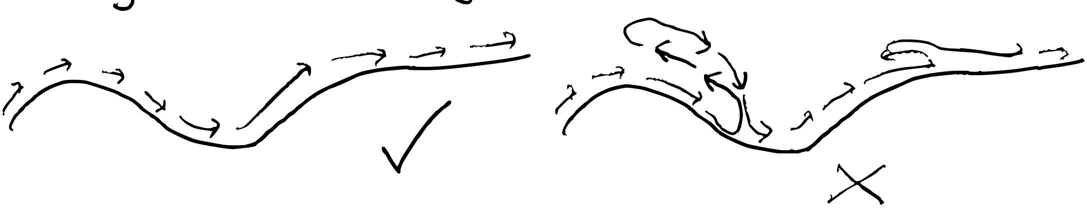
In expressing a curve with some parameterization,

it is important to not change the direction.

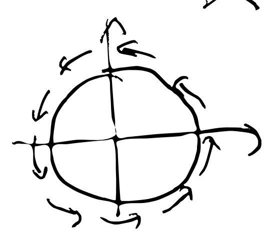
Namely, given the path



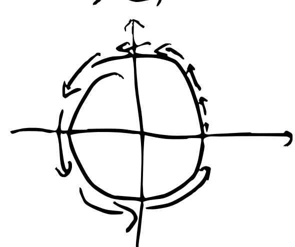
you are allowed to go in either direction with any speed, but you shouldn't change your direction once you start moving.



Thus, $\langle r\cos t, r\sin t\rangle$ $0 \le t \le 2\pi$



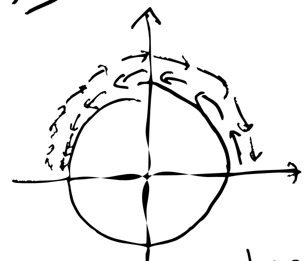
$\langle r\cos(t^2), r\sin(t^2)\rangle$ $0 \le t \le \sqrt{2\pi}$



are ok,

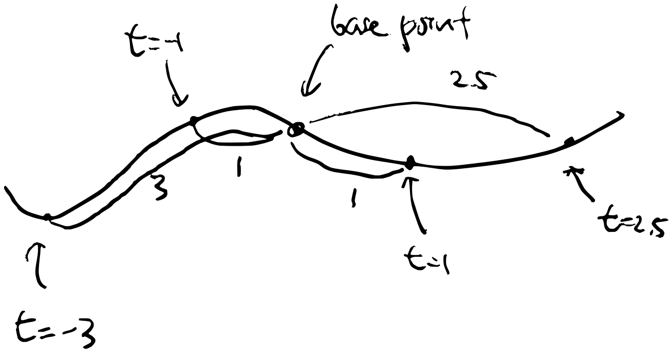
but $\langle r\cos(\pi\sin(t)), r\sin(\pi\sin(t))\rangle$

$0 \le t \le \pi$



is not OK, even though this parametrization has the same start/end points.

It will be useful to have a standard way of parametrizing a curve. We can use the notion of arclength:



Namely, the time corresponds to the arclength from the base point! This is called the arclength parametrization.

Definition $\vec{r}(t)$ is an arclength parametrization if $|\vec{r}\,'(t)| = 1$.

3 Steps of finding arclength parametrization

Step 1. Find the arclength function $l(t)$.

If the start point is $t=a$, $l(t)$ should be

$$\boxed{\begin{array}{ll} l(t)=\int_a^t |\vec{r}\,'(t)|\,dt & (\text{increasing } t \text{ direction}) \\ l(t)=\int_t^a |\vec{r}\,'(t)|\,dt & (\text{decreasing } t \text{ direction}) \end{array}}$$

Step 2 Express t in terms of l.

Step 3 Put the expression from Step2 to $\vec{r}(t)$ to obtain the arclength parametrization $\vec{r}(l)$.

Example Find the arclength parametrization of $\vec{r}(t)=\langle \cos(t^3), \sin(t^3), t^3\rangle$ starting from $(1,0,0)$ to the direction of increasing t.

Sol Note that $(1,0,0)$ corresponds to $t=0$,

$$\vec{r}\,'(t)=\langle -3t^2\sin(t^3), 3t^2\cos(t^3), 3t^2\rangle$$

$$|\vec{r}\,'(t)|=\sqrt{(-3t^2\sin(t^3))^2+(3t^2\cos(t^3))^2+(3t^2)^2}$$

$$=\sqrt{9t^4(\sin^2(t^3)+\cos^2(t^3)+1)}$$

$$=\sqrt{18t^4}=3\sqrt{2}\,t^2$$

Step1 The arclength function is

$$\ell(t) = \int_0^t |\vec{r}'(t)|dt = \int_0^t 3\sqrt{2}t^2 dt = \sqrt{2}t^3.$$

Step 2 Since $\ell = \sqrt{2}t^3$, $t = \sqrt[3]{\frac{\ell}{\sqrt{2}}}$.

Step3 We plug this back into $\vec{r}(t)$ and get

$$\boxed{\vec{r}(\ell) = \left\langle \cos\left(\frac{\ell}{\sqrt{2}}\right), \sin\left(\frac{\ell}{\sqrt{2}}\right), \frac{\ell}{\sqrt{2}} \right\rangle}$$

Sanity check:

$$\vec{r}'(\ell) = \left\langle -\frac{\sin\left(\frac{\ell}{\sqrt{2}}\right)}{\sqrt{2}}, \frac{\cos\left(\frac{\ell}{\sqrt{2}}\right)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$|\vec{r}'(\ell)| = \sqrt{\left(-\frac{\sin\left(\frac{\ell}{\sqrt{2}}\right)}{\sqrt{2}}\right)^2 + \left(\frac{\cos\left(\frac{\ell}{\sqrt{2}}\right)}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{2}\left(\sin^2\left(\frac{\ell}{\sqrt{2}}\right) + \cos^2\left(\frac{\ell}{\sqrt{2}}\right) + 1\right)} = \sqrt{1} = 1.$$

Understanding the motion

As mentioned before, the notion of vectors came from physics. What governs how things move is Newton's Second Law,

Force = Mass × Acceleration,

or

$\vec{F}(t) = m\vec{a}(t)$

Mass is the weight, like 1 kg (we use the standard unit, kg, m, .. in math, physics).

Acceleration $\vec{a}(t)$ is the double derivative of the position vector,

$\vec{a}(t) = \vec{r}\,''(t)$. ($\vec{v}(t) = \vec{r}\,'(t)$ is on the other hand the velocity)

What is Force then? It's a physics notion, and it accounts for all physical interactions happening on the setting. For example, on Earth, there is the gravity (gravitational force) which is $\vec{F}_{gravity}(t) = -9.8m\vec{k}$. Here, 9.8 is the number that came out of experiments, and $-(\ \ )\vec{k}$ means the gravity always pulls everything downwards.

<u>Example</u> Suppose at the origin you throw a 1 kg ball at the speed of 1m/s to the positive x-direction in the angle of $60°$. Suppose the gravity is the only force that acts on the ball. What is the trajectory of the ball until it reaches the ground again?

<u>Solution</u> We have

$$m\vec{a}(t) = \vec{F}(t) = -9.8\, m\vec{k}, \quad \text{or}$$

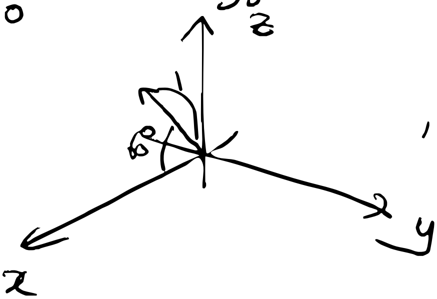
$$\vec{r}''(t) = \vec{a}(t) = -9.8\vec{k}.$$

We let the initial position to be the origin $(0,0,0)$.

By the fundamental theorem of calculus,

$$\vec{r}'(t) - \vec{r}'(0) = \int_0^t \vec{r}''(u)\,du = \int_0^t -9.8\vec{k}\,du = -9.8t\vec{k}$$

Since $\vec{r}'(0)$ is



, its components are the rectangular coordinates for the spherical coordinate point $(\rho, \theta, \phi) = (1, 0, 30°) = (1, 0, \frac{\pi}{6})$.

So $(x, y, z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

$$= \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$$

$\Rightarrow \vec{r}'(0) = \langle \frac{1}{2}, 0, \frac{\sqrt{3}}{2} \rangle \Rightarrow \vec{r}'(t) = \vec{r}'(0) - 9.8t\vec{k} = \langle \frac{1}{2}, 0, \frac{\sqrt{3}}{2} - 9.8t \rangle.$

By the fundamental theorem of calculus again,

$$\vec{r}(t) - \vec{r}(0) = \int_0^t \vec{r}'(u)\,du = \int_0^t \left\langle \tfrac{1}{2}, 0, \tfrac{\sqrt{3}}{2} - 9.8u \right\rangle du$$

$$= \left\langle \tfrac{1}{2}t, 0, \tfrac{\sqrt{3}}{2}t - 4.9t^2 \right\rangle.$$

Since the start point at time $t=0$ is the origin, $\vec{r}(0) = \langle 0,0,0 \rangle$,

so $\vec{r}(t) = \left\langle \tfrac{1}{2}t, 0, \tfrac{\sqrt{3}}{2}t - 4.9t^2 \right\rangle$. is the trajectory.

In particular, it is a parabola. With only gravity, the trajectory is always a parabola!